A general framework for studying rainbow configurations

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Abstract

Given a coloring of a set, classical Ramsey theory looks for various configurations within a color class. Rainbow configurations, also called anti-Ramsey configurations, are configurations that occur across distinct color classes. We present some very general results about the types of colorings that will guarantee various types of rainbow configurations in finite and infinite settings, as well as several illustrative corollaries.

1 Introduction

Classical Ramsey problems typically involve partitioning an ambient set up into disjoint subsets called color classes, then looking for conditions under which a given configuration will be present in one of the color classes. One canonical example is Schur's Theorem, which says that if you color the natural numbers with a finite number of colors, then there must be a color class with a triple of the form $(x, y, x+y)$.

Here, we consider so-called anti-Ramsey or *rainbow* problems. Rainbow problems have been studied in different contexts; see [1] and [6] and the references therein for some arithmetic rainbow results, and see [2] and [7] and the references therein for results in graph theory. We begin with some basic definitions.

Definition 1.1. *A* ***coloring*** *of X is a function $f : X \to \mathcal{C}$ for some set $\mathcal{C}$ of colors; the preimages $\{f^{-1}(i)\}$ are the* ***color classes*** *of the coloring. Notice that the color classes form a partition of X.*

Definition 1.2. *A* ***rainbow configuration*** *is a set $(x_1, x_2, \ldots, x_k) \in X^k$ such that $x_1, x_2, \ldots, x_k$ all belong to distinct color classes.*

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1.1 Results

In this paper, we prove some very general results on what kinds of colorings will guarantee the existence of various types of rainbow configurations. Our first main result is very general, so we first give an illustrative corollary, which can be seen as a rainbow version of Schur's Theorem. We refer the reader to [6] and [1] for related results.

Theorem 1.3. *Suppose we have a coloring of a finite abelian group, G. If no color class has size $\geq \frac{2}{27}|G|$ then there must be a rainbow triple of the form $(x, y, x + y)$.*

Theorem 1.3 is a corollary of a more general result, which we now state as our first main theorem. In this statement, E is the set of k-tuples of elements of X that form whatever configuration we are concerned with (such as $(x, y, x+y)$ in Theorem 1.3). The quantity M depends on how many of these k-tuples share a pair of coordinates (in Theorem 1.3, each pair of coordinates uniquely determines a triple).

Theorem 1.4. *Let X be a finite set of size n, and $E \subset X^k$ be a set of k-tuples, with the property that there exist constants $M_{i,j}$, such that for any*

$$x = (x_1, x_2, \ldots, x_k) \in E,$$

we have that $|\{y \in E : y_i = x_i, y_j = x_j\}| \leq M_{i,j}$. Define

$$M = \sum_{i<j} M_{i,j}.$$

If no color class has size $\geq Cn$, where $|E| \geq Dn^2$, and $C < \frac{2D}{9M}$, then there must be a rainbow k-tuple in E.

We now list a few other applications of Theorem 1.4. The next result was inspired by the corresponding Ramsey problem: which colorings of various rings admit monochromatic quadruples of the form $(x, y, x + y, xy)$? Note that this result involves both addition and multiplication. We state a similar result in the negative. That is, if there are no rainbow configurations, then we must have a color class that is too large. See [3, 4, 5, 8] for background and related Ramsey type problems. We let $\mathbb{F}_q$ denote the finite field of q elements.

Corollary 1.5. *If we color $\mathbb{F}_q$ such that each color class has size $< \frac{(2-o(1))q}{63}$, then there must be a rainbow quadruple of the form $(x, y, x + y, xy)$.*

Following this, we have a result guaranteeing the existence of long rainbow arithmetic progressions in a wide class of finite abelian groups. See [10] and the references contained therein ([1] in particular), for more on rainbow arithmetic progressions.

Corollary 1.6. *If we color a finite (additive) abelian group G, with no (non-identity) element of order $< k$, then if there are no rainbow k-arithmetic progressions, at least one color class has size $\geq \frac{2}{9\binom{k}{2}}|G|$.*

We also include this application of Theorem 1.4 with an ambient set that is not even a group.

Corollary 1.7. *If we color $[1..n]$ such that there are no rainbow triples of the form $(x, y, x+y)$, then at least one color class has size $\geq \frac{1-o(1)}{27}n$.*

Under some slightly stronger conditions, we can estimate how many rainbow configurations must be present.

Theorem 1.8. *Suppose the conditions of Theorem 1.4 are satisfied, but are strengthened so that no color class has size $\geq Cpn$ for some $p \in (0, 1]$. Then there are at least $(1-p)Dn^2$ rainbow elements of E.*

A more general statement is the following theorem, which works on finite measure spaces in place of finite sets, but with special types of colorings. However, to deal with infinite sets, we need to introduce some restrictions on the types of colorings and measures that we can use.

Definition 1.9. *Let (X, μ) be a measure space. We call a coloring of X **tractable** if every color class is a measurable subset of X and there are only at most countably many color classes.*

The theorems below hold in more generality than this; however, the restrictions involved (and the ensuing measures ν_t) seem to be more complicated. While special cases may still hold for colorings that are not tractable, Example 3.2, given later, illustrates why such restrictions are needed in general.

Theorem 1.10. *Let (X, μ) be a finite measure space, $n = \mu(X) < \infty$. Suppose we have $E \subset X^k$, $k \geq 3$, with the property that there exist constants $M_{i,j}$ such that for any*

$$x = (x_1, x_2, \ldots, x_k) \in E,$$

we have that $|\{y \in E : y_i = x_i, y_j = x_j\}| \leq M_{i,j}$. Define

$$M = \sum_{i<j} M_{i,j}.$$

$$m_{ij}(E) = \iint |\{y \in E : x_i = y_i, x_j = y_j\}| dx_i dx_j$$

$$m(E) = \sum_{i<j} m_{ij}(E)$$

If we have a tractable coloring where no color class has measure $\geq Cn$, with $m(E) \geq D\binom{k}{2}n^2$, and $C < \frac{2D}{9M}$, then there must be a rainbow element in $x \in E$.

Note that we can prove Theorem 1.4 by applying Theorem 1.10 with μ as an appropriate counting measure. To see how Theorem 1.10 works with more general measures, we prove a continuous version of Theorem 1.3.

Corollary 1.11. *Fix a probability measure μ on the unit circle in the complex plane. If we color the circle with at least 14 equally-sized μ-measurable color classes, there must be a rainbow triple of the form (x, y, xy) in the circle.*

Our final example application along these lines is geometric. This is analogous to the result due to the second listed author in [9] in vector spaces over finite fields.

Corollary 1.12. *If we split any square into at least 104 equally-sized Lebesgue measurable color classes, then it must contain three points x_1, x_2, x_3 such that $|x_1 - x_2| = |x_2 - x_3| = |x_1 - x_3|$ and the color classes of the points are distinct (i.e. a rainbow equilateral triangle).*

Finally, we present an even more general result involving so-called "subrainbow" configurations. These are configurations which may not be strictly rainbow, but have no color represented too many times.

Definition 1.13. *Let X have a coloring; let $x \in X^k$. Then x is* **w-subrainbow** *if x has no w components of the same color.*

Our usual definition of rainbow is here called 2-subrainbow, because any rainbow k-tuple has no 2 elements of the same color. We now introduce a technical definition, which we will motivate more in Subsection 3.2.1. For now, suffice it to say that it quantifies of the an amount of freedom inherent in various configurations.

Definition 1.14. *Let (X, μ) be a finite measure space. Let $E \subset X^k$, and fix a $t \in [2 \ldots k-1]$. For any subset $S \subset [1..k]$, with $|S| = t$, write S as $\{s_1, s_2, \ldots, s_t\}$ and define $M(S)$ to be*

$$\sup_{(y_1,\ldots,y_k)\in E} |\{x \in E : x_{s_j} = y_{s_j}\}|.$$

Then we say that the ***t-bound*** *of E is $\sum_S M(S)$, and we call this value M_t so long as it is finite. Similarly, define $m(S)$ to be*

$$\int \cdots \int |y \in E : y_{s_i} = x_{s_i}, i \in [1..t]| dx_{s_1} \ldots dx_{s_2}$$

and define $m = \sum_S m_S$.

We pause to note that we can actually do a little better than the bound on C in the following result, depending on the various parameters, but the bound given always holds.

Theorem 1.15. *Let (X, μ) be a finite measure space, and define $n = \mu(X)$. Let $E \subset X^k$ be measurable, and let the t-bound of E be M, and suppose $m \geq D\binom{k}{t}n^t$ and $2 \leq w \leq t < k$. Then for any tractable coloring of X, if $\mu(A) \leq Cn$ for all color classes A, E must contain a w-subrainbow element, as long as*

$$C^{w-1} < \frac{2^{w-1}D}{3^w M\binom{t}{w}}.$$

To illustrate the difference between this and the preceding theorems, here is a corollary:

Corollary 1.16. *Let G be a finite abelian group, $2, 3 \nmid |G|$, with some coloring. Then:*

- *if no color class has size $\geq \frac{1}{135}|G|$, there must be a rainbow quintuple of the form $(x, y, z, x + y + z, x + 2y + 3z)$.*
- *if no color class has size $\geq \sqrt{\frac{2}{135}}|G|$, there must be a 3-subrainbow quintuple of the form $(x, y, z, x + y + z, x + 2y + 3z)$.*

1.2 Outline of the rest of the paper

We will start by proving Theorem 1.4, then use it to prove its corollaries, Theorem 1.3, Corollaries 1.5, 1.6, and 1.7, and Theorem 1.8 in Section 2. In the next section, we will explain how to modify the proof of Theorem 1.4 to obtain Theorem 1.10, the infinite setting. We will then apply Theorem 1.10 to prove Corollaries 1.11 and 1.12. Finally, we will finish by showing how to modify the above arguments further to prove Theorem 1.15, which deals with subrainbow configurations, and use it to prove Corollary 1.16.

2 Theorem 1.4 and some applications

2.1 Proof of Theorem 1.4

Proof. The basic idea of this proof will be to assume that we have no rainbow configuration, then find that there must be some large color class. To do this, we will merge color classes together to make a uniform count. Note that merging color classes can destroy but not create rainbow configurations. So if we began without a rainbow configuration, then merging classes will not create one.

Fix $C > 0$ to be determined later. Now, proceed to greedily merge the smallest two color classes pairwise until every class has size between $(1/2)Cn$ and $(3/2)Cn$. Let s denote the number of color classes after this merging.

There are $|E|$ k-tuples in the set E. Fix a color i, and let n_i denote the number of elements from X in color class i. Recall that M bounds how many k-tuples from E share (at least) a pair of coordinates. So there are at most Mn_i^2 k-tuples in E with at least two elements with color i. Now, if there are no rainbow k-tuples then every k-tuple in E must have at least two coordinates of the same color, so

$$|E| \leq M \sum_{i=1}^{s} n_i^2 \leq M \sum_{i=1}^{s} \left(\frac{3}{2}Cn\right)^2 \leq \frac{9M}{4} sC^2 n^2,$$

where we used the assumption that $n_i \leq (3/2)Cn$ for all i. So we have that

$$s \geq \frac{4}{9M} \frac{1}{C^2} \frac{|E|}{n^2} \geq \frac{4D}{9M} \frac{1}{C^2}.$$

But every class has size at least $(1/2)Cn$, which, since they are all disjoint, implies that X has at least

$$\frac{1}{2}Cn \cdot \frac{4D}{9M} C^{-2} = \frac{2D}{9M} \frac{n}{C} > n$$

elements, a contradiction. □

2.2 Corollaries of Theorem 1.4

Here, we prove Theorem 1.3, Corollaries 1.5, 1.6, and 1.7, and Theorem 1.8. These illustrate how the various quantities arise.

2.2.1 Proof of Theorem 1.3

Proof. We will apply Theorem 1.4 with $X = G$ and $k = 3$. The set $E \subset X^3$ will be the set of triples of G of the form $(x, y, x + y)$. Now, $M_{1,2}$ will be the number of different triples in G that can share the first two coordinates. But any pair of first and second coordinates, x and y, will uniquely determine the third coordinate, $x + y$. There may be other triples in X^3 that share the first two coordinates, but only one of them will be in E, so $M_{1,2} = 1$. Notice that any pair of elements x and $x + y$ will uniquely determine y, so $M_{1,3} = 1$. Similarly, $M_{2,3}$ will be 1, so

$$M = \sum_{i<j} M_{i,j} = M_{1,2} + M_{1,3} + M_{2,3} = 3.$$

All that is left is for us to apply Theorem 1.4 is to get a value for D. So we need to see how big E is in terms of X. As each pair of elements, x and y, from G generates a distinct triple $(x, y, x + y)$ in E, we see that $|E| = n^2$, so $D = 1$. Plugging everything into Theorem 1.4 guarantees the existence of a rainbow triple of the form $(x, y, x + y)$ for any coloring where each color class is smaller than Cn, where

$$C < \frac{2D}{9M} = \frac{2}{27}.$$

□

2.2.2 Proof of Corollary 1.5

Proof. Similar to the proof of Theorem 1.3, we will set $X = \mathbb{F}_q$, find a set of quadruples, E, and estimate the constants M and D to plug into Theorem 1.4. Initially, we would start with all quadruples of the form $(x, y, x + y, xy)$ as E, but we drop all quadruples with $xy = 0$, because given $xy = x = 0$ there are still many possible quadruples, and a quadruple with $x = xy$ would necessarily be nonrainbow. So $M_{1,4}$ would be too large to get an effective bound. This leaves the number of possible quadruples that comprise our set E to be

$$q^2 - 2q + 1 = (1 - o(1))q^2.$$

Now, $|\mathbb{F}_q| = q$, so $n = q$. Since $E = (1 - o(1))n^2$, we have that $D = (1 - o(1))$.

Knowing any two of $x, y, x+y$ clearly fixes the rest of the tuple, and knowing xy and either x or y does the same (as $xy \neq 0$). Therefore $M_{i,j} = 1$ for all of the $M_{i,j}$ except $M_{3,4}$. If we know $x + y$ and xy then x and y must be roots of the polynomial $t^2 - (x+y)t + (xy)$, of which there are at most two (so there are at most two quadruples, since we can change the order of x and y). This gives us that $M_{3,4} = 2$, and summing this with the other $M_{i,j}$ gives $M = 7$.

When we put these values into Theorem 1.4, we get that there must be a rainbow quadruple of the form $(x, y, x + y, xy)$ if all of the color classes are smaller than Cq, for

$$C < \frac{2D}{9M} = \frac{2 - o(1)}{63}.$$

□

2.2.3 Proof of Corollary 1.6

Proof. Again, we will find a suitable set E, then compute the corresponding values of M and D. Since our ambient group is G, we have that $n = |G|$. We set E to be the set of all ordered k-tuples whose elements form an ordered k-term arithmetic progression:

$$E := \{(z, z + x, \dots, z + (k - 1)x) : z, x \in G\}.$$

Note that in E, we could have two elements that consist of the same group elements, but in distinct orders. Each of the k-term arithmetic progressions above will be distinct, giving us n^2 distinct elements in E, one for every pair of elements, $(z, x) \in G^2$. So $D = 1$.

To get a handle on M, we need to see how often two k-term arithmetic progressions can have two elements in the same spot (e.g., they have the same fifth element and same ninth element). So suppose that the k-term arithmetic progressions generated in E by (z, x) and (z', x') share the same elements at the slots numbered a and b, for some distinct $a, b \in [0..(k - 1)]$. That is,

$$z + ax = z' + ax' \text{ and } z + bx = z' + bx'.$$

This gives $ax - ax' = z' - z = bx - bx'$. Then $a(x - x') = b(x - x')$, meaning

$$(a - b)(x - x') = 0;$$

so $x - x'$ has order $\leq |a - b|$, meaning $x = x'$ (since $|a - b| < k$), which implies that the two progressions are identical. Thus all the $M_{i,j}$ are 1, and there are $\binom{k}{2}$ of them. So $M = \binom{k}{2}$, $D = 1$ and the result follows. ☐

2.2.4 Proof of Corollary 1.7

Proof. This runs essentially the same way as the proof of Theorem 1.3, with $M = 3$. However, there is a slightly different calculation for D. Note that if we choose $x = c$, there are $n - c$ possible choices for y such that $x + y \in [1 \ldots n]$. Therefore:

$$|E| = \sum_{c=1}^{n} n - c = \binom{n}{2} = \left(\frac{1}{2} - o(1)\right) n^2,$$

giving $D = \frac{1}{2} - o(1)$, and we apply Theorem 1.4. ☐

2.2.5 Proof of Theorem 1.8

Proof. Using Theorem 1.4, we see that there must be a rainbow element of E. Thus we can remove it from E, and we can keep finding rainbow elements as long as $|E| > Dpn^2$. So there are at least $Dn^2 - Dpn^2 = (1 - p)Dn^2$ rainbow elements of E. ☐

3 More general settings

Here, we show how the proof of Theorem 1.4 can be modified to handle much more general settings.

3.1 Theorem 1.10

3.1.1 Proof of Theorem 1.10

Proof. As before, merging color classes can destroy but not create rainbow elements; so if we merge classes and still have a rainbow element, there must have been a rainbow element beforehand. We will use Zorn's lemma to find a nice coloring. Define a coloring λ to be a **merging** of a coloring λ' if for every x_1 and x_2 of the same color under λ', x_1 and x_2 are the same color under λ (which is equivalent to saying that every color class in λ is a union of color classes in λ'). Since we have countably many colors, merging color classes is the same as taking at most countable unions of measurable sets, so color classes of mergings will still be measurable.

Call our original coloring λ_0, and let Λ be the set of all colorings that are mergings of λ_0 and have no color class of size $> Cn$. Then we can place a partial ordering on Λ by $\lambda_1 \preceq \lambda_2$ when λ_2 is a merging of λ_1. Suppose $\{\lambda_i\}_{i \in I}$

is a nonempty chain in Λ; we define λ by the rule that x has the same color as x' in λ if and only if x has the same color as x' in some λ_i. Then $\lambda \succeq \lambda_0$. For contradiction, suppose λ contains a color class A of measure $Cn + \epsilon$. Then there exist finitely many nonempty color classes $A_1, \ldots, A_m$ of λ_0 such that $A_i \subset A$ and $\sum \mu(A_i) > Cn$. Pick $x_i \in A_i$. Then by definition of λ, there exist $j_1, \ldots, j_m \in I$ such that x_1 is the same color as x_i in λ_{j_i} (since they're the same color in λ). Define j to be the maximum of all of these $j_1, \ldots, j_m \in I$, which exists as $\{\lambda_i\}_{i \in I}$ is a nonempty chain. But then all the x_i are the same color in λ_j, meaning λ_j has a color class of size at least $\sum \mu(A_i) > Cn$, and contradicting $\lambda_j \in \Lambda$. So every chain in Λ has an upper bound in Λ, meaning Zorn's lemma applies ($\Lambda \neq \emptyset$ since $\lambda_0 \in \Lambda$); hence there exists a maximal coloring λ.

Now, λ can have only at most one color class of measure $< (1/2)Cn$ (since if it had two, we could merge them and get a class of measure $< Cn$, contradicting maximality). If we have such a color class, then we can merge it with one of the other ones so that all classes have measure between $(1/2)Cn$ and $(3/2)Cn$ and if we don't then we already have this bound. Note that since every color class has measure $\geq (1/2)Cn$, there are finitely many color classes; let the number of color classes be s.

If there are no rainbow k-tuples, then

$$m(E) \leq \binom{k}{2} M \sum_{A \in \lambda} \mu(A)^2 \leq M \binom{k}{2} \sum_{A \in \lambda} \left(\frac{3}{2} Cn\right)^2 \leq \frac{9M}{4} \binom{k}{2} s C^2 n^2,$$

so $s \geq \frac{4}{9M} C^{-2} \frac{m(E)}{\binom{k}{2} n^2} \geq \frac{4D}{9M} C^{-2}$. But every class has size at least $(1/2)Cn$, which, since they are all disjoint, implies X has total measure at least

$$\frac{1}{2} Cn \cdot \frac{4D}{9M} C^{-2} = \frac{2D}{9M} \frac{n}{C} > n,$$

which is a contradiction. □

3.1.2 Proof of Corollary 1.11

Proof. In what follows, we will assume that all color classes have the same μ-measure. Call the unit circle in the complex plane X. By definition, (X, μ) is a finite measure space with $\mu(X) = 1$. Our set E will be the set of all triples of the form $(x, y, xy) : x, y \in X$. Now, since any pair, $(x, y) \in X^2$, uniquely determines a triple in E, and all triples in E can be so determined, we have that $m(E) = 3\mu(X) \cdot \mu(X) = 3$. So $D = 1$. Also, as in the proof of Theorem 1.3, we will see that $M = 3$. So, by Theorem 1.10, we are guaranteed a rainbow triple of the form (x, y, xy), as long as we have a tractable coloring where no color class has μ-measure $\geq Cn$, where

$$\frac{2D}{9M} = \frac{2}{9 \cdot 3} = \frac{2}{27} \geq C.$$

Since each color class has μ-measure $(1/14) < (2/27)$, we satisfy the hypotheses of Theorem 1.10. □

3.1.3 Pathological colorings

These examples serve to motivate the restriction of tractability in Theorem 1.10. The first is a simple example to show that the existence of a rainbow triple of the form $(x, y, x+y)$ need not exist in a coloring of $\mathbb{R}$ with an infinite number of color classes.

Example 3.1. *Color $\mathbb{R}$ by the smallest power of two in the denominator. That is, color class 0 will be all of the rational numbers with an odd denominator, when written in lowest terms, color class 1 will be all numbers with a single power of 2 in the denominator, when written in lowest terms, and so on. Finally, put the irrationals in their own color class, called ∞. Now suppose that x is in color class i and y is in color class j, where i and j could be nonnegative integers or ∞. Then $x+y$ will live in the color class $\max(i, j)$, and we will have no rainbow triples of the form $(x, y, x+y)$.*

If we drop the assumption that all color classes are measurable then we can get some paradoxical behavior. The following example illustrates this.

Example 3.2. *Let $X = \mathbb{R}$ under any measure; let $\{x_i\}_{i\in I}$ be a well-ordered basis for $\mathbb{R}$ over $\mathbb{Q}$. Then define A_j to be the set of linear combinations $\sum_{i\in I} c_i x_i$ with all but finitely many c_i equal to zero, and $j = \min(i : c_i \neq 0)$. Since the x_i form a basis, this is indeed a coloring of $\mathbb{R}$, but for any $a_i \in A_i$, $a_{i'} \in A_{i'}$, $i \neq i'$, $a + a', a - a', a' - a \in A_{\min(i,i')}$. So there are in fact no rainbow triples of the form $(x, y, x+y)$ in this coloring.*

3.1.4 Proof of Corollary 1.12

Proof. Since the desired property is independent of rotation or scaling, we can assume $X = [0, 1]^2$. For any fixed x_1, any x_2 which is closer to x_1 than x_1 is to the boundary of X will result in 2 equilateral triangles contained in X since each of the points equidistant from x_1 and x_2 will be contained in X; the measure of the set of x_2 satisfying this is equal to the area of the largest possible circle centered at x_1 contained in X. By symmetry it suffices to consider the eighth of the square defined by $x \in [0, 1/2], y \in [0, x]$ (and for all these points the closest side is the bottom). So we get a total measure of

$$\begin{aligned} 2 \cdot 8 \int_0^{1/2} \int_0^x \pi y^2 dy dx &= 16\pi \int_0^{1/2} \frac{x^3}{3} dx \\ &= \frac{16}{3}\pi \frac{(1/2)^4}{4} \\ &= \frac{\pi}{12} \end{aligned}$$

which means $m(E) = 3\frac{\pi}{12}$ so $D = \frac{\pi}{12}$.

Applying Theorem 1.10 with $M = 6$ and $D = \frac{\pi}{12}$, we get that

$$\left\lceil \frac{1}{C} \right\rceil = \left\lceil \frac{9M}{2D} \right\rceil = \left\lceil \frac{324}{\pi} \right\rceil = 104$$

colors of equal measure suffice. $\square$

3.2 Theorem 1.15

3.2.1 Explaining t-bounds

In the results up to this point, we have been using M, which is the t-bound with $t = 2$. This is why the $M_{i,j}$ measured when distinct configurations could share exactly 2 coordinates. For example, if you look back at the previous proofs, the 2-bound of the set of additive triples of the form $(x, y, x + y)$ is 3; the 2-bound on the set of k-length arithmetic progressions in $\mathbb{Z}_p$ is $\binom{k}{2}$ as long as $k \ll p$; the 2-bound on sets of the form $(x, y, x + y, xy)$ in $\mathbb{Z}_p$ is 7.

For higher order t bounds, we will be considering how often there are t coordinates shared by a pair of distinct configurations. For example, suppose we were considering k-point configurations in $\mathbb{R}^d$. Then the 3-bound would be measuring how many of the k-point configurations being considered could share 3 points. These higher-order t-bounds will result from configuration spaces where more than two elements are required in order to determine a tuple up to some small multiplicity. For example, if we are looking for rainbow quadruples $(x, y, z, x + y + z)$ in some abelian group, we will be interested in the 3-bound.

3.2.2 Proof of Theorem 1.15

Proof. By the same Zorn's Lemma merging argument as before, we can assume that every color class has size between $\frac{1}{2}Cn$ and $\frac{3}{2}Cn$. As a result of this merging process there are only finitely many colors; call this number s. If no element in E is w-subrainbow, then every element of E has at least w elements from at least one color. For any fixed color A, the ν_t-measure of $\{x \in E$ with at least w elements of $A\}$ is at most $M\binom{t}{w}\mu(A)^w n^{t-w}$. So, summing over all the color classes we get

$$
\begin{aligned}
Dn^t \leq \nu_t(E) &\leq M\binom{t}{w}\sum_A \mu(A)^w n^{t-w} \\
&\leq \binom{t}{w} M \left(\frac{3}{2}Cn\right)^w \sum_A n^{t-w} \\
&\leq \binom{t}{w} M \left(\frac{3}{2}C\right)^w n^w \cdot sn^{t-w} \\
&= \binom{t}{w} M \left(\frac{3}{2}C\right)^w sn^t
\end{aligned}
$$

which means

$$
s \geq \frac{2^w D}{3^w C^w \binom{t}{w} M}.
$$

Since each color class has measure at least $\frac{1}{2}Cn$, and they are all disjoint,

$$n = \mu(X) \geq \frac{2^w D}{3^w C^w M\binom{t}{w}} \cdot \frac{1}{2}Cn > n,$$

which is a contradiction, so there must be a w-subrainbow element. □

3.2.3 Proof of Corollary 1.16

Proof. Given any three of $(x, y, z, x + y + z, x + 2y + 3z)$, the other two are uniquely determined (since 2 and 3 do not divide $|G|$ for each $g \in G$ there are unique h_2, h_3 such that $2h_2 = 3h_3 = g$, and we can solve for x, y, z whilst only dividing out by these two numbers). So $M_3 = \binom{5}{3} = 10$; since there is a unique quintuple for every $(x, y, z) \in G^3$, there are $|G|^3$ total quintuples, so $D = 1$. $t = 3$, and the two cases correspond to $w = 2$ and $w = 3$ respectively. Plugging in to the statement of Theorem 1.15 we get

$$C < \frac{2 \cdot 1}{3^2 \cdot 10 \cdot \binom{3}{2}} = \frac{1}{135}$$

in the first case and

$$C^2 < \frac{2^2 \cdot 1}{3^3 \cdot 10 \cdot \binom{3}{3}} = \frac{2}{135}$$

$$C < \sqrt{\frac{2}{135}}$$

in the second case. □

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